An inductive proof of Straub's q-analogue of Ljunggren's congruence*

Bo Ning†

Department of Applied Mathematics, Northwestern Polytechnical University,

Xi'an, Shaanxi 710072, P.R. China

Abstract

Recently, Straub gave an interesting q-analogue of a binomial congruence of Ljunggren. In this note we give an inductive proof of his result.

Keywords: q-analogue; q-congruence; binomial coefficient; Ljunggren's congruence

AMS Subject Classification (2000): 05A10, 11A07, 11B65

1 Introduction

q-Series has been proved to be a challenging and interesting area in number theory. For a basic introduction to q-series and a wonderful survey paper, see [3, Chapter 10] and [4], respectively. In particular, q-analogues of a lot of classical congruences have been studies by several authors. We refer the readers to [2, 5, 6, 13, 15, 17, 19]. For a detailed talk about q-congruences, we refer to Pan's Ph.D thesis [12].

As shown in [3], we use $[n]_q := 1+q+q^2+\ldots+q^{n-1} = \frac{1-q^n}{1-q}$, $[n]!_q := [n]_q[n-1]_q\cdots[1]_q$ and $\binom{n}{k}_q := \frac{[n]!_q}{[k]!_q[n-k]!_q}$ to denote the usual q-analogues of numbers, factorials and binomial

*Supported by the Doctorate Foundation of Northwestern Polytechnical University (cx201326).

†E-mail address: ningbo_math84@mail.nwpu.edu.cn (B. Ning)

coefficients, respectively. It is easy to see that the usual numbers, factorials and binomial coefficients can be obtained as $q = 1$.

The classical Lucas' congruence [10] tells us how to compute a binomial coefficient modulo a prime.

Theorem 1 (Lucas, [10])**.** *For any prime p, we can determine $\binom{n}{m} \pmod p$ from the base p expansions of n and m. Specially, if $n = \sum_{i=0}^{t} b_i p^i$ and $m = \sum_{i=0}^{t} c_i p^i$ where $0 \le b_i, c_i < p$, then*

$$\binom{n}{m} \equiv \prod_{i=0}^{t} \binom{b_i}{c_i} \pmod p. \tag{1}$$

In particular, when $n = kp$ and $m = sp$, (1) implies that $\binom{kp}{sp} \equiv \binom{k}{s} \pmod p$. For the case that a binomial coefficient modulo a prime power, Ljunggren [9] gave an interesting extension in 1952.

Theorem 2 (Ljunggren, [9])**.** *For any prime $p \ge 5$ and nonnegative integers k, s,*

$$\binom{kp}{sp} \equiv \binom{k}{s} \pmod{p^3}. \tag{2}$$

Recently, Straub [19] gave a q-analogue of Ljunggren's binomial congruence (2).

Theorem 3 (Straub, [19])**.** *For any prime $p \ge 5$ and nonnegative integers k, s,*

$$\binom{kp}{sp}_q \equiv \binom{k}{s}_{q^{p^2}} - \binom{k}{s+1}\binom{s+1}{2}\frac{p^2-1}{12}(q^p-1)^2 \pmod{[p]_q^3}. \tag{3}$$

Note that Straub's proof largely depends on the method in [6]. In this note we give an inductive proof of Straub's result.

Remark 1. For q-binomial coefficients, there is a combinatorial interpretation in terms of areas under lattice paths due to Pólya, see [16, Vol.4, p.444]. In [18, Chapter 1, Problem 6 (d)], Stanley gave a combinatorial proof of Theorem 2. Maybe it is interesting to find a combinatorial proof of Theorem 3.

2 An inductive proof of Theorem 3

The following two results are well-known (see [1, (3.3.10)] and [7, 11]).

Lemma 1. (The q-Chu-Vandermonde-formula) *For nonnegative integers m, n and h,*

$$\sum_{k=0}^{h} \binom{n}{k}_q \binom{m}{h-k}_q = \binom{m+n}{h}_q .$$

Lemma 2. (The q-Lucas-Theorem) *For any prime p and nonnegative integers a, b, r and s such that $0 \le b, s \le p-1$,*

$$\binom{ap+b}{rp+s}_q \equiv \binom{a}{r}\binom{b}{s}_q \pmod{[p]_q}.$$

The next Lemma ([19, Lemma 5]) is a big step of Straub's proof. We first give a new proof of this lemma.

Lemma 3. *For any prime $p \ge 5$,*

$$\binom{2p}{p}_q \equiv [2]_{q^{p^2}} - \frac{p^2-1}{12}(q^p-1)^2 \pmod{[p]_q^3}. \tag{4}$$

Proof. By the q-Chu-Vandermonde-formula,

$$\binom{2p}{p}_q = \sum_{i=0}^{p} \binom{p}{i}_q^2 q^{i^2} = 1 + q^{p^2} + \sum_{i=1}^{p-1} \binom{p}{i}_q^2 q^{i^2} = [2]_{q^{p^2}} + \sum_{i=1}^{p-1} \binom{p}{i}_q^2 q^{i^2}.$$

Thus we need only show that $\sum_{i=1}^{p-1} \binom{p}{i}_q^2 q^{i^2}$ is congruence (mod $[p]_q{}^3$) to $-\frac{p^2-1}{12}(q^p-1)^2$. Since

$$\binom{p}{i}_q^2 q^{i^2} = (\frac{[p]!_q}{[i]!_q[p-i]!_q})^2 q^{i^2} = [p]_q^2(\frac{[p-1]!_q}{[i]!_q[p-i]!_q})^2 q^{i^2},$$

we need only show that $\sum_{i=1}^{p-1}(\frac{[p-1]!_q}{[i]!_q[p-i]!_q})^2 q^{i^2}$ is congruence (mod $[p]_q$) to $-\frac{p^2-1}{12}(1-q)^2$. Noting that $q^p \equiv 1 \ (\mathrm{mod}[p]_q)$, we have

$$\begin{aligned}
&(\tfrac{[p-1]!_q}{[i]!_q[p-i]!_q})^2 q^{i^2} \\
&= (\tfrac{(1-q^{p-1})(1-q^{p-2})\cdots(1-q^{p-i+1})}{(1-q)(1-q^2)\cdots(1-q^i)})^2 q^{i^2}(1-q)^2 \\
&= (\tfrac{(q-q^p)(q^2-q^p)\cdots(q^{i-1}-q^p)}{(1-q)(1-q^2)\cdots(1-q^i)})^2 q^i (1-q)^2
\end{aligned}$$

$$\equiv (\tfrac{(q-1)(q^2-1)\cdots(q^{i-1}-1)}{(1-q)(1-q^2)\cdots(1-q^i)})^2 q^i (1-q)^2 \pmod{[p]_q}$$
$$= \tfrac{q^i(1-q)^2}{(1-q^i)^2},$$

and it implies that $\sum_{i=1}^{p-1}(\frac{[p-1]!_q}{[i]!_q[p-i]_q})^2 q^{i^2}$ is congruence (mod $[p]_q$) to $\sum_{i=1}^{p-1}\frac{q^i(1-q)^2}{(1-q^i)^2}$. Hence we are done if $\sum_{i=1}^{p-1}\frac{q^i}{(1-q^i)^2}$ is congruence (mod $[p]_q$) to $-\frac{p^2-1}{12}$. In fact, this is a deformation of Lemma 2 in [17] due to Shi and Pan. The proof is complete. □

As a second step of an inductive proof of Theorem 3, the following lemma is needed.

Lemma 4. *For any prime $p \geq 5$,*

$$\binom{kp}{p}_q \equiv \binom{k}{1}_{q^{p^2}} - \binom{k}{2}\frac{p^2-1}{12}(q^p-1)^2 \pmod{[p]_q^3}. \tag{5}$$

Proof. For a given integer k, if $k = 1$, the proposition is trivially true. If $k = 2$, it can be deduced from Lemma 3. Now we assume that $k \geq 3$. By the q-Chu-Vandermonde formula,

$$\begin{aligned}
L &= \tbinom{kp}{p}_q \\
&= \textstyle\sum_{i=0}^{p}\binom{(k-1)p}{p-i}_q\binom{p}{i}_q q^{i((k-2)p+i)} \\
&= \tbinom{(k-1)p}{p}_q + q^{(k-1)p^2} + \textstyle\sum_{i=1}^{p-1}\binom{(k-1)p}{p-i}_q\binom{p}{i}_q q^{i((k-2)p+i)} \\
&= \tbinom{(k-1)p}{p}_q + q^{(k-1)p^2} + \textstyle\sum_{i=1}^{p-1}\binom{p}{i}_q q^{i((k-2)p+i)}\sum_{j=0}^{p-i}\binom{(k-2)p}{p-i-j}_q\binom{p}{j}_q q^{j((k-3)p+i+j)} \\
&= \tbinom{(k-1)p}{p}_q + q^{(k-1)p^2} + \textstyle\sum_{i=1}^{p-1}\binom{p}{i}_q\binom{(k-2)p}{p-i}_q q^{i((k-2)p+i)} + \sum_{i=1}^{p-1}\binom{p}{i}_q\binom{p}{p-i}_q q^{p^2(k-2)+i^2} \\
&+ \textstyle\sum_{i=1}^{p-1}\sum_{j=1}^{p-i-1}\binom{p}{i}_q\binom{(k-2)p}{p-i-j}_q\binom{p}{j}_q q^{i((k-2)p+i)+j((k-3)p+i+j)}.
\end{aligned}$$

Now let $s(i,j) = i((k-2)p+i) + j((k-3)p+i+j)$ and let

$$\begin{aligned}
L_1 &= \tbinom{(k-1)p}{p}_q + q^{(k-1)p^2}, \\
L_2 &= \textstyle\sum_{i=1}^{p-1}\binom{p}{i}_q\binom{(k-2)p}{p-i}_q q^{i((k-2)p+i)}, \\
L_3 &= \textstyle\sum_{i=1}^{p-1}\binom{p}{i}_q\binom{p}{p-i}_q q^{p^2(k-2)+i^2}, \\
L_4 &= \textstyle\sum_{i=1}^{p-1}\sum_{j=1}^{p-i-1}\binom{p}{i}_q\binom{(k-2)p}{p-i-j}_q\binom{p}{j}_q q^{s(i,j)}.
\end{aligned}$$

By the induction hypothesis,

$$\begin{aligned}
L_1 &\equiv \tbinom{k-1}{1}_{q^{p^2}} + q^{(k-1)p^2} - \tbinom{k-1}{2}\tfrac{p^2-1}{12}(q^p-1)^2 \pmod{[p]_q^3} \\
&= \tbinom{k}{1}_{q^{p^2}} - \tbinom{k-1}{2}\tfrac{p^2-1}{12}(q^p-1)^2.
\end{aligned}$$

On the other hand, by the q-Lucas-Theorem, for $1 \le i \le p-1$, $\binom{p}{i}_q \equiv \binom{(k-2)p}{p-i}_q \equiv 0 \pmod{[p]_q}$, and we also have $q^{i((k-2)p+i)} \equiv q^{i((k-3)p+i)} \pmod{[p]_q}$. By the induction hypothesis,

$$\begin{aligned} L_2 &\equiv \sum_{i=1}^{p-1} \binom{p}{i}_q \binom{(k-2)p}{p-i}_q q^{i((k-3)p+i)} \pmod{[p]_q^3} \\ &= \binom{(k-1)p}{p}_q - \binom{(k-2)p}{p}_q - q^{(k-2)p^2} \\ &\equiv (\binom{k-1}{1}_{q^{p^2}} - \binom{k-2}{1}_{q^{p^2}} - q^{(k-2)p^2}) - (\binom{k-1}{2} - \binom{k-2}{2}) \tfrac{p^2-1}{12}(q^p-1)^2 \pmod{[p]_q^3} \\ &= -(k-2)\tfrac{p^2-1}{12}(q^p-1)^2. \end{aligned}$$

Similarly, we have

$$\begin{aligned} L_3 &= \sum_{i=1}^{p-1} \binom{p}{i}_q \binom{p}{p-i}_q q^{p^2(k-2)+i^2} \\ &\equiv \sum_{i=1}^{p-1} \binom{p}{i}_q \binom{p}{p-i}_q q^{i^2} \pmod{[p]_q^3} \\ &= \binom{2p}{p}_q - 1 - q^{p^2} \\ &\equiv [2]_{q^{p^2}} - 1 - q^{p^2} - \tfrac{p^2-1}{12}(q^p-1)^2 \pmod{[p]_q^3} \\ &= -\tfrac{p^2-1}{12}(q^p-1)^2 \end{aligned}$$

and

$$L_4 \equiv 0 \pmod{[p]_q^3}.$$

Thus, we have

$$\begin{aligned} L &= L_1 + L_2 + L_3 + L_4 \\ &\equiv \binom{k}{1}_{q^{p^2}} - \binom{k-1}{2}\tfrac{p^2-1}{12}(q^p-1)^2 - (k-2)\tfrac{p^2-1}{12}(q^p-1)^2 - \tfrac{p^2-1}{12}(q^p-1)^2 \pmod{[p]_q^3} \\ &= \binom{k}{1}_{q^{p^2}} - \binom{k}{2}\tfrac{p^2-1}{12}(q^p-1)^2. \end{aligned}$$

The proof is complete. □

Remark 2. Motivated by Wilson's theorem which states that $(p-1)! \equiv -1 \pmod{p}$ if p is a prime, Wolstenholme [20] proved that for primes $p \ge 5$,

$$\binom{2p-1}{p-1} \equiv 1 \pmod{p^3}. \tag{6}$$

Later, Glaisher [8] improved Wolstenholme's result (6) by proving that if p is a prime ≥ 5, then

$$\binom{mp+p-1}{p-1} \equiv 1 \pmod{p^3}. \tag{7}$$

Note that (4) and (5) can be considered as q-analogues of Wolstenholme's congruence (6) and Glaisher's congruence (7), respectively.

Proof of Theorem 3. We use induction on s and k to give a proof. For a given integer k, if $s = 0$, it is trivially true. If $s = 1$, it can deduced from Lemma 4. If $k \leq s$, the result is also right. Now we assume that $k > s \geq 2$ and for a fixed s, we induct on k. By the q-Chu-Vandermonde formula,

$$\begin{aligned} L &= \tbinom{kp}{sp}_q \\ &= \textstyle\sum_{i=0}^{p} \tbinom{(k-1)p}{sp-i}_q \tbinom{p}{i}_q q^{i((k-s-1)p+i)} \\ &= \tbinom{(k-1)p}{sp}_q + \tbinom{(k-1)p}{(s-1)p}_q q^{(k-s)p^2} + \textstyle\sum_{i=1}^{p-1} \tbinom{(k-1)p}{sp-i}_q \tbinom{p}{i}_q q^{i((k-s-1)p+i)} \\ &= \tbinom{(k-1)p}{sp}_q + \tbinom{(k-1)p}{(s-1)p}_q q^{(k-s)p^2} + \textstyle\sum_{i=1}^{p-1} \tbinom{p}{i}_q q^{i((k-s-1)p+i)} \sum_{j=0}^{p} \tbinom{(k-2)p}{sp-i-j}_q \tbinom{p}{j}_q q^{j((k-2-s)p+i+j)} \\ &= \tbinom{(k-1)p}{sp}_q + \tbinom{(k-1)p}{(s-1)p}_q q^{(k-s)p^2} + \textstyle\sum_{i=1}^{p-1} \tbinom{p}{i}_q \tbinom{(k-2)p}{sp-i}_q q^{i((k-s-1)p+i)} + \sum_{i=1}^{p-1} \tbinom{p}{i}_q \tbinom{(k-2)p}{(s-1)p-i}_q \cdot \\ &\quad q^{(p+i)((k-1-s)p+i)} + \textstyle\sum_{i=1}^{p-1} \sum_{j=1}^{p-1} \tbinom{p}{i}_q \tbinom{(k-2)p}{sp-i-j}_q \tbinom{p}{j}_q q^{i((k-1-s)p+i)+j((k-2-s)p+i+j)}. \end{aligned}$$

Now let $s(i,j) = i((k-1-s)p+i) + j((k-2-s)p+i+j)$ and let

$$\begin{aligned} L_1 &= \tbinom{(k-1)p}{sp}_q + \tbinom{(k-1)p}{(s-1)p}_q q^{(k-s)p^2}, \\ L_2 &= \textstyle\sum_{i=1}^{p-1} \tbinom{p}{i}_q \tbinom{(k-2)p}{sp-i}_q q^{i((k-s-1)p+i)}, \\ L_3 &= \textstyle\sum_{i=1}^{p-1} \tbinom{p}{i}_q \tbinom{(k-2)p}{(s-1)p-i}_q q^{(p+i)((k-1-s)p+i)}, \\ L_4 &= \textstyle\sum_{i=1}^{p-1} \sum_{j=1}^{p-1} \tbinom{p}{i}_q \tbinom{(k-2)p}{sp-i-j}_q \tbinom{p}{j}_q q^{s(i,j)}. \end{aligned}$$

By the induction hypothesis,

$$\begin{aligned} L_1 &\equiv \tbinom{k-1}{s}_{q^{p^2}} + \tbinom{k-1}{s-1}_{q^{p^2}} q^{(k-s)p^2} - \{\tbinom{k-1}{s+1}\tbinom{s+1}{2} + \tbinom{k-1}{s}\tbinom{s}{2} q^{(k-s)p^2}\} \tfrac{(p^2-1)(1-q)^2}{12} [p]_q^2 \pmod{[p]_q^3} \\ &= \tbinom{k}{s}_{q^{p^2}} - \{\tbinom{k-1}{s+1}\tbinom{s+1}{2} + \tbinom{k-1}{s}\tbinom{s}{2}\} \tfrac{(p^2-1)(1-q)^2}{12} [p]_q^2. \end{aligned}$$

On the other hand, for $1 \leq i \leq p-1$, $\binom{p}{i}_q \equiv \binom{(k-2)p}{sp-i}_q \equiv 0 \pmod{[p]_q}$ and $q^{i((k-s-1)p+i)} \equiv$

$q^{i((k-s-2)p+i)} \pmod{[p]_q}$. By the induction hypothesis,

$$\begin{aligned}
L_2 &\equiv \sum_{i=1}^{p-1} \binom{p}{i}_q \binom{(k-2)p}{sp-i}_q q^{i((k-s-2)p+i)} \pmod{[p]_q^3} \\
&= \{\sum_{i=0}^{p} \binom{p}{i}_q \binom{(k-2)p}{sp-i}_q q^{i((k-s-2)p+i)}\} - \binom{(k-2)p}{sp}_q - \binom{(k-2)p}{(s-1)p}_q q^{p^2(k-s-1)} \\
&= \binom{(k-1)p}{sp}_q - \binom{(k-2)p}{sp}_q - \binom{(k-2)p}{(s-1)p}_q q^{p^2(k-s-1)} \\
&\equiv \{\binom{k-1}{s}_{q^{p^2}} - \binom{k-2}{s}_{q^{p^2}} - \binom{k-2}{s-1}_{q^{p^2}} q^{p^2(k-s-1)}\} - \{\binom{k-1}{s+1}\binom{s+1}{2} - \binom{k-2}{s+1}\binom{s+1}{2} - \binom{k-2}{s}\binom{s}{2}\} \cdot \\
&\quad \frac{(p^2-1)(1-q)^2}{12}[p]_q^2 \pmod{[p]_q^3} \\
&= -\{\binom{k-1}{s+1}\binom{s+1}{2} - \binom{k-2}{s+1}\binom{s+1}{2} - \binom{k-2}{s}\binom{s}{2}\} \cdot \frac{(p^2-1)(1-q)^2}{12}[p]_q^2 \\
&= -\{\binom{k-2}{s}s\} \cdot \frac{(p^2-1)(1-q)^2}{12}[p]_q^2.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
L_3 &\equiv \sum_{i=1}^{p-1} \binom{p}{i}_q \binom{(k-2)p}{(s-1)p-i}_q q^{i((k-1-s)p+i)} \pmod{[p]_q^3} \\
&= \{\sum_{i=0}^{p} \binom{p}{i}_q \binom{(k-2)p}{(s-1)p-i}_q q^{i((k-1-s)p+i)}\} - \binom{(k-2)p}{(s-1)p}_q - \binom{(k-2)p}{(s-2)p}_q q^{p^2(k-s)} \\
&= \binom{(k-1)p}{(s-1)p}_q - \binom{(k-2)p}{(s-1)p}_q - \binom{(k-2)p}{(s-2)p}_q q^{p^2(k-s)} \\
&\equiv \{\binom{k-1}{s-1}_{q^{p^2}} - \binom{k-2}{s-1}_{q^{p^2}} - \binom{k-2}{s-2}_{q^{p^2}} q^{p^2(k-s)}\} - \{\binom{k-1}{s}\binom{s}{2} - \binom{k-2}{s}\binom{s}{2} - \binom{k-2}{s-1}\binom{s-1}{2}\} \cdot \\
&\quad \frac{(p^2-1)(1-q)^2}{12}[p]_q^2 \pmod{[p]_q^3} \\
&= -\{\binom{k-1}{s}\binom{s}{2} - \binom{k-2}{s}\binom{s}{2} - \binom{k-2}{s-1}\binom{s-1}{2}\} \cdot \frac{(p^2-1)(1-q)^2}{12}[p]_q^2 \pmod{[p]_q^3} \\
&= -\{\binom{k-2}{s-1}(s-1)\} \cdot \frac{(p^2-1)(1-q)^2}{12}[p]_q^2
\end{aligned}$$

and

$$\begin{aligned}
L_4 &= \sum_{i=1}^{p-1}\sum_{j=1}^{p-1} \binom{p}{i}_q \binom{(k-2)p}{sp-i-j}_q \binom{p}{j}_q q^{i((k-1-s)p+i)+j((k-2-s)p+i+j)} \\
&\equiv \sum_{i+j=p, i\ge 1, j\ge 1} \binom{p}{i}_q \binom{p}{j}_q \binom{(k-2)p}{(s-1)p}_q q^{i(p-j)} \pmod{[p]_q^3} \\
&= \{\sum_{i+j=p} \binom{p}{i}_q \binom{p}{j}_q \binom{(k-2)p}{(s-1)p}_q q^{i(p-j)}\} - \binom{(k-2)p}{(s-1)p}_q (1+q^{p^2}) \\
&= \binom{2p}{p}_p \cdot \binom{(k-2)p}{(s-1)p}_q - \binom{(k-2)p}{(s-1)p}_q (1+q^{p^2}) \\
&\equiv \{[2]_{q^{p^2}} - 1 - q^{p^2}\} \cdot \binom{(k-2)p}{(s-1)p}_q - \binom{k-2}{s-1}_{q^{p^2}} \cdot \frac{(p^2-1)(1-q)^2}{12}[p]_q^2 \pmod{[p]_q^3} \\
&\equiv -\binom{k-2}{s-1} \cdot \frac{(p^2-1)(1-q)^2}{12}[p]_q^2 \pmod{[p]_q^3}.
\end{aligned}$$

Note that

$$\binom{k}{s+1}\binom{s+1}{2} = \binom{k-1}{s+1}\binom{s+1}{2} + \binom{k-1}{s}\binom{s}{2} + \binom{k-2}{s}s + \binom{k-2}{s-1}(s-1) + \binom{k-2}{s-1}.$$

Thus we have

$$\begin{aligned} L &= L_1 + L_2 + L_3 + L_4 \\ &\equiv \tbinom{k}{s}_{q^{p^2}} - \tbinom{k}{s+1}\tbinom{s+1}{2} \cdot \tfrac{(p^2-1)(1-q)^2}{12}[p]_q^2 \pmod{[p]_q^3}. \end{aligned}$$

The proof is complete. □

3 Another q-analogue of Ljunggren's congruence

Glaisher's congruence (7) can be written as

$$(mp+1)(mp+2)\ldots(mp+p-1) \equiv (p-1)! \pmod{p^3}. \tag{8}$$

In 1999, Andrews [2] gave a q-analogue (9) of Glaisher's congruence (8): If p is an odd prime and $m \geq 1$, then

$$\frac{(q^{mp+1};q)_{p-1} - q^{mp(p-1)/2}(q;q)_{p-1}}{(1-q^{(m+1)p})(1-q^{mp})} \equiv \frac{(p^2-1)p}{24} \pmod{[p]_q}. \tag{9}$$

Recently, with the help of Andrews' q-analogue (9), Pan [14, Lemma 3.1] got a general q-analogue of Ljunggren's congruence (2). The following q-analogue can be deduced from his result.

Theorem 4. *For any prime $p \geq 5$ and nonnegative integers k, s,*

$$\binom{kp}{sp}_q \equiv q^{(k-s)s\binom{p}{2}} \cdot (\binom{k}{s}_{q^p} + k\binom{k}{s+1}\binom{s+1}{2}\frac{p^2-1}{12}(q^p-1)^2) \pmod{[p]_q^3}. \tag{10}$$

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